Lecture 23. The characteristic polynomial

Def The characteristic polynomial of a square matrix A is

$$\boxed{p_A(\lambda) := \det(A - \lambda I)}$$

Thm The eigenvalues of a square matrix A are precisely the roots of the characteristic polynomial $p_A(\lambda)$.

pf A has an eigenvalue $\lambda \iff \det(A-\lambda I) = 0 \iff p_A(\lambda) = 0$

Prop Given a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we have

$$\boxed{p_A(\lambda) = \lambda^2 - (a+d)\lambda + (ad-bc)}$$

pf $p_A(\lambda) = \det(A-\lambda I) = \det\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$

$$= (a-\lambda)(d-\lambda) - bc = ad - a\lambda - d\lambda + \lambda^2 - bc$$

$$= \lambda^2 - (a+d)\lambda + (ad-bc)$$

Prop If a square matrix A is triangular, its eigenvalues are given by its diagonal entries.

e.g. $A = \begin{bmatrix} 2 & 1 & 5 \\ 0 & 3 & 7 \\ 0 & 0 & 4 \end{bmatrix}$

$$\Longrightarrow p_A(\lambda) = \det(A-\lambda I) = \det\begin{bmatrix} 2-\lambda & 1 & 5 \\ 0 & 3-\lambda & 7 \\ 0 & 0 & 4-\lambda \end{bmatrix} = (2-\lambda)(3-\lambda)(4-\lambda)$$

$\Longrightarrow p_A(\lambda)$ has roots at $\lambda = \underbrace{2, 3, 4}_{\text{diagonal entries}}$

Note (1) If A is an $n\times n$ matrix, $p_A(\lambda)$ has degree n

$\Longrightarrow$ A has at most n distinct eigenvalues

(2) For square matrices of large size, characteristic polynomials are not useful for finding eigenvalues. However, characteristic polynomials have many other applications, some of which we will discuss in Lecture 26.

Def Let α be an eigenvalue of a square matrix A

(1) Its algebraic multiplicity is the exponent of the factor $\lambda-\alpha$ in the factorization of $p_A(\lambda)$.

(2) Its geometric multiplicity is the dimension of $\underbrace{\mathrm{Nul}(A-\lambda I)}_{\alpha\text{-eigenspace}}$.

e.g. $A=\begin{bmatrix}3&1\\0&3\end{bmatrix} \Longrightarrow p_A(\lambda)=\lambda^2-6\lambda+9=(\lambda-3)^{2}$

$\Longrightarrow$ $\lambda=3$ is an eigenvalue with algebraic multiplicity 2

$A-3I=\begin{bmatrix}0&1\\0&0\end{bmatrix} \Longrightarrow \dim(\mathrm{Nul}(A-3I))=1$

$\Longrightarrow$ $\lambda=3$ has geometric multiplicity 1

Prop Let α be an eigenvalue of a square matrix A

(1) Its geometric multiplicity is at most its algebraic multiplicity.

(2) If its algebraic multiplicity is 1, so is its geometric multiplicity.

Ex Find all eigenvalues of each matrix.

(1) $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

Sol $p_A(\lambda) = \lambda^2 - (2+2)\lambda + (2\cdot 2 - 1\cdot 1) = \lambda^2 - 4\lambda + 3 = (\lambda-1)(\lambda-3)$

$\Rightarrow$ A has eigenvalues $\boxed{\lambda = 1, 3}$

(2) $B = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$

Sol $p_B(\lambda) = \lambda^2 - (1+3)\lambda + (1\cdot 3 - (-1)\cdot 1) = \lambda^2 - 4\lambda + 4 = (\lambda-2)^2$

$\Rightarrow$ B has a unique eigenvalue $\boxed{\lambda = 2}$

(3) $C = \begin{bmatrix} 5 & 7 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

Sol Since C is triangular, its eigenvalues are given by the diagonal entries $\boxed{1, 2, 5}$

(4) $D = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & 6 \\ 0 & -1 & 1 \end{bmatrix}$

Sol $p_D(\lambda) = \det(D - \lambda I) = \det \begin{bmatrix} 1-\lambda & 2 & 0 \\ 3 & 2-\lambda & 6 \\ 0 & -1 & 1-\lambda \end{bmatrix}$

$= (1-\lambda)\det \begin{bmatrix} 2-\lambda & 6 \\ -1 & 1-\lambda \end{bmatrix} - 3\det \begin{bmatrix} 2 & 0 \\ -1 & 1-\lambda \end{bmatrix}$

$= (1-\lambda)\left[(2-\lambda)(1-\lambda) + 6\right] - 3\cdot 2(1-\lambda)$

$= (1-\lambda)\left[(2-\lambda)(1-\lambda) + \cancel{6} - \cancel{6}\right]$

$= (1-\lambda)^2(2-\lambda)$

$\Rightarrow$ D has eigenvalues $\boxed{\lambda = 1, 2}$

Ex Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

(1) Find all eigenvalues of A with their algebraic multiplicities.

Sol $$p_A(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 0 & 2 \\ 0 & 1-\lambda & 3 \\ 0 & 0 & 2-\lambda \end{bmatrix} = (1-\lambda)^2(2-\lambda)$$

Hence the eigenvalues of A are

$$\boxed{\begin{cases} \lambda = 1 \text{ with algebraic multiplicity } 2 \\ \lambda = 2 \text{ with algebraic multiplicity } 1 \end{cases}}$$

Note In fact, for a triangular matrix, the algebraic multiplicity of an eigenvalue λ equals the number of times λ appears on the diagonal.

(2) For each eigenvalue of A, find its geometric multiplicity.

Sol The geometric multiplicity of an eigenvalue λ is $\dim(\mathrm{Nul}(A - \lambda I))$.

$$\lambda = 1: \quad A - I = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow \mathrm{RREF}(A - I) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

col 1, col 2 with no leading 1's

$$\Rightarrow \dim(\mathrm{Nul}(A - I)) = 2$$

$$\lambda = 2: \quad A - 2I = \begin{bmatrix} -1 & 0 & 2 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathrm{RREF}(A - 2I) = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

col 3 with no leading 1's

$$\Rightarrow \dim(\mathrm{Nul}(A - 2I)) = 1$$

Hence the geometric multiplicity is $\boxed{2 \text{ for } \lambda = 1 \text{ and } 1 \text{ for } \lambda = 2}$